LABORATOIRE DE PHYSIQUE THEORIQUE DE LA MATIERE CONDENSEE
L.P.T.M.C.

K. A. Penson
Professeur

Tour 13, 5ème étage, Boîte Courrier 121
4, Place Jussieu, F 75252 Paris Cedex 05, France
Tél. : +33 (0)1 44.27.72.33
Fax : +33 (0)1 44.27.51.00
email: penson@lptl.jussieu.fr

UMR 7600

To whom it may concern

REFEREE's REPORT

Subject : PhD Thesis of mgr. ing. Grzegorz SIUDEM,

entitled :« **Applications of Combinatorial Methods to the Study of State Spaces in Some Models of Statistical Physics »**

submitted to the Faculty of Physics of Warsaw Polytechnics, Warsaw, Poland - June 2017.

I was assigned to express my opinion about the above Thesis and in the following you will find my remarks which I hope will be useful for your judgement.

This Thesis is a rather voluminous work in Theoretical Physics dedicated to very timely subjects lying in the basis of Statistical Physics of discrete systems. The Thesis is of an interdisciplinary character as the tools used and developed in it heavily relay on techniques of combinatorial nature. The author has displayed a noteworthy initiative to explore less known mathematical approaches and to effectively apply them to important physical problems. As such, the Thesis can be classified to belong to the Combinatorial Physics, which is in fact a rapidly developing research trend now.

The present Referee welcomes the works in this direction, as for a some time now, he was himself involved in a research in similar spirit.

The Thesis was written under direction of Prof. A. Fronczak, whose recent articles (these are refs. [56], [57], [58] in the Thesis) appear to be the source of inspiration for Mr. Siudem, who apparently very effectively employed and pushed forward Fronczak's methods. There is a sense of complicity and collaboration between the PhD candidate Siudem and his Thesis Advisor Prof. Fronczak which transpires from the reading his Thesis, a very satisfactory circumstance indeed.

The Thesis is composed of four Chapters, the final summary, and three Appendices. The second Appendix B contains many details of proofs omitted in the main text. At the end of the Thesis one finds an exhaustive reference list (containing 169 items) that I will comment upon below. The results presented in the Thesis have appeared in two articles, and in two articles in course of publication.

I will present in short my impressions from reading this Thesis.

The first Chapter is 69 pages long, contains the motivation, the derivation of the Darwin-Fowler method and serves as a detailed introduction to the problems treated in the following Chapters. The author discusses at length the notions of statistical ensembles with rich references to a number of fundamental handbooks of Statistical Mechanics. Already at this stage a short discussion of the Bell polynomials is presented which indicates their utility to calculate the statistical sums. An inspiring Remark 6 relates the Darwin-Fowler method with the asymptotics of Bell polynomials which justifies their role in further parts of the Thesis. Subsequently follows an about 25 pages long paragraph on the Ising model. The author succeeds in presenting the essential features of this model which will be relevant for futher discussions. They include the review of classical and also less known references, low- and high-temperature expansions, with their graphical visualisations via polyominos and the lattice animals, the Onsager solution and its variants, the method of series expansions, and cluster expansions, as well as a discussion of combinatorial approaches. Mr. Siudem made an considerable effort to dig out recent reviews and more exotic approaches to the Ising model so that this paragraph is clear, complete and self-contained. I had a real pleasure reading it. The Chapter is concluded with a nice discussion of Bell polynomials which stresses their potential utility in Statistical Mechanics.

Chapter 2 is devoted to a presentation of the model of ideal gas of clusters treated with Fronczak's formalism which employs the Bell polynomials introduced before. The main result is the Theorem 3, Eq.(2.7) that is inductively proven in Appendix B.5, following [145]. Furthermore, the 1D lattice gas is treated using the transfer matrix method, which when compared with Fronczak's methods, furnishes interesting relations with the Stirling numbers of the first kind, thus eminently combinatorial constructs.

Chapter 3 treats a number of various subjects related to the 1D and 2D Ising models with the special emphasis on utility and explicit applications of the Bell polynomials. As an example I quote Eq. (3.11), which is the low-temperature expansion for the infinite square lattice. The demonstration of Eq.(3.11) is quite clever : it uses both logarithmic and conventional Bell polynomials , s. Appendix B.6.1. The change of symmetry of the underlying lattices implies the use of exact solutions obtained by Wannier and Domb (for the triangular and the hexagonal lattices). The high- and low-temperature expansions for these cases were obtained by rather involved manipulations using the Bell polynomials. They are summarized in Eqs. (3.43), (3.44) and (3.45), and are backed by the details that are meticulously exposed in Appendices B.7 and B.8. In all these formulas one encounters the expressions for the number of walks of length l, $S(l)$. They turn out to be related to certain integrals of trigonometric functions. This had motivated the author to analyse that kind of problem separately.

Chapter 4 treats these observed relations which lead to the Hypothesis 2 on p.115 : it predicts the existence of integral representations of $S(l)$ in terms of trigonometric integrals for a large class of lattices. The Hypothesis requires a careful definition of admissible lattices and a clean formulation of appropriate walks. All of that has been done in Sections 4.1 and 4.2, with the final result Eq.(4.3), which gives a general counting of walks in question. Some amusing extensions of phenomenological nature are also given. They consist in *postulating* a specific form of integrand and then analyzing resulting expansion in relation to given integer sequences. In this way the graph character was « forgotten » and only the algebraic input in form of a chosen integrand remains. Nevertheless several known sequences from the OIES were obtained in this manner ! I consider this aspect quite intriguing and I gather that Mr. Siudem and his collaborators are preparing an extended version of this Chapter, to be published very soon. Hopefully, the Hypothesis will become the Theorem !

As the last element of the Thesis we find the summary of specific applications of the Bell polynomials in the Thesis, as well as the list of open problems. The Appendices that follow are very carefully formulated and they clarify missing elements in the text.

My verdict is :

I find this Thesis extremely well written, with a great care for precision and clarity. It contains a number of new results which are worth of pursuing further. It is *not* a set of publications stapled together (as can be frequently seen in many places recently). On the contrary, it is an extended work cross-referencing Siudem's results and publications, but also containing additional useful elements of synthetic nature. The author has a complete command of many tools of Statistical Mechanics. But he is also introducing and courageously applying novel tools (Bell polynomials) which seemingly have an enormous potential of applications, and should inspire new research in this and related fields. An important part of the Thesis is a truly excellent reference list :
the present Referee found dozens of useful references that escaped his attention till now. This also proves Mr. Siudem's curiosity and personal interest in the research done before.

I thought that it would be useful to situate this Thesis in view of growing interplay between

the Theoretical Physics and Combinatorics ; not surprisingly, Siudem, Fronczak and Fronczak in [148] do it very well themselves and I quote them here:
« In this paper, we use some ideas and formulas, which originate from combinatorics, to get the exact expressions for the coefficients. And although our result is important in itself, it is also a pretext to draw physicists' attention to the progress made in recent years in (enumerative) combinatorics…. ».
It goes without saying that I very strongly support the attitude expressed in this statement.

In view of the above remarks my very strong recommendation is that Mr. Grzegorz SIUDEM be awarded the title of PhD of the Warsaw Polytechnics, Warsaw, Poland.

Penson

K. A. Penson

Professor, Sorbonne Université, Paris VI, France

Paris, May 20 2017